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Fractals in Physics: Squig Clusters, Diffusions, Fractal Measures, and the Unicity of Fractal Dimensionality

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The three topics discussed in this paper are largely independent. Part 1: Fractal "squig clusters" are introduced, and it is shown that their properties can match to a remarkable extent those of percolation clusters at criticality. Physics on these new geometric shapes should prove tractable. As background, the author's theories of squig intervals and squig trees are reviewed, and restated in more versatile form. Part 2: The notion of "latent" fractal dimensionality is introduced and motivated by the desire to simplify the algebra of dimensionality. Scaling noises are touched upon. A common formalism is presented for three forms of anomalous diffusion: the ant in the fractal labyrinth, fractional Brownian motion, and Lévy stable motion. The fractal dimensionalities common to diverse shapes generated by diffusion are given, in Table I, as functions of the latent dimensionalities of the support of the motion and of the diffusion itself. Part 3: It is argued that every fractal point set has a unique fractal dimensionality, but it is pointed out that many fractals involve diverse combinations of many fractal point sets. Such is, in particular, the case for fractal measures and for fractal graphs, often called hierarchical lattices. The fractal measures that the author had introduced in the early 1970s are described, including new developments.

KEY WORDS: Diffusion; fractals; percolation clusters; squig models.

0. INTRODUCTION: IN PRAISE OF EXPLICIT GEOMETRY

It is impossible to survey the present role of fractals in statistical physics because this role is changing and expanding too rapidly, and there is no need to summarize the basic ideas because they have become widely known

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and used. The present paper is, instead, an informal discussion of a number of individual topics of very active current interest, as listed in the abstract. Some were actually part of my presentation at Statphys 15, while others respond to questions that were asked there.

A theme that links these topics is that the usefulness of explicit and constructive geometry should again be recognized by physicists. Blind analytic manipulation is not enough. Physics is of course rife with quantities that originated in geometry; unfortunately, most are used only in analytic relationships. Thus, after I had conceived fractal geometry,[1-3] and used it to study mountains and other visible and well-describable natural shapes in natural space, its first effect upon physics was to provide yet another *analytic* quantity to evaluate: the fractal dimensionality D. The disregard of geometry is the root of many confusions. For example, the various disputes that arise about the value of D, with claims that D can take either of two or more values for the "same fractal," all spring from the neglect of geometry. In some cases, this ambiguity merely confirms or reveals that several *different* shapes (i.e., compact point sets) are in fact inherent in the *same* physical problem. In other cases, some of these values concern statistical *populations* of shapes, whereas dimensionality should carefully be kept as a property of *sample* shapes. If explicit geometry is kept in mind throughout, such errors should be avoided, and the field's progress should be swifter and smoother.

Scaling geometry is a genuine geometric counterpart to scaling analysis. The contention that mathematicians have reduced geometry to analysis, deserves nothing but scorn.

A small proportion of this paper is a straight exposition of parts of Ref. 3, rearranged very differently; much is a substantial restatement or is completely new; and substantial portions represent a change of mind. FGN will be a self-explanatory acronym for Ref. 3. Since the paper deals with several separate topics, a certain amount of repetition was left in to make it easier to scan.

This paper will continue in this journal's issue devoted to the Gaithersburg Conference on Fractals in the Physical Sciences.

1. PART ONE: FRACTAL SQUIGS AND A NEW MODEL OF PERCOLATION CLUSTERS

Many shapes in physics, e.g., self-avoiding random walks and percolation clusters, are usefully approximated by random scaling fractals. However, the construction of these fractals is not only not recursive, but is very indirect. They are the limits, as the cells in a lattice are downsized to 0, of broken lines drawn on these lattices (see FGN, Chapter 36). This is one reason why the above shapes are difficult to simulate and difficult to

handle analytically. A recursive fractal model of percolation clusters is proposed in FGN, Chapter 13[4]; it has proven useful, but it is not random.

To meet the related challenges of modeling the self-avoiding random walks and the self-avoiding river trees, I introduced in 1978[5] a very different new family of fractals, which are both recursively constructed and random; I now call them "fractal squig intervals and trees." Their theory is sketched in FGN, Chapter 24, and in Section 1.2 below.

But the underlying idea goes beyond "intervals," and I have recently extended it to percolation, by introducing "fractal squig clusters," as described in Section 1.1. Then Section 1.2 examines one by one the "menu" of basic ideas that underlie the notion of squig.

1.1. A Fractal Squig Model for Percolation Clusters in the Plane

1.1.1. A New Construction. Fractal dimensionality takes the value $\log 8/\log 3 \sim 1.8928$ for the Sierpinski triadic carpet, and many authors (Refs. 6 and 7 and FGN, Chapter 24 gives the references) find essentially the same value for the percolation clusters in the plane. Nevertheless, the carpet is not at all suitable as a model of the clusters. First of all, the carpet's topology is all wrong: it has no dangling bonds, and it is infinitely ramified (FGN, Chapter 14; see also Ref. 8), while percolation clusters are finitely ramified (see Section 1.1.7, also Refs. 8 to 10). Moreover, define a "ring" as a portion of a cluster that is multiply connected (between any two points there are at least two distinct paths) and is maximal (it does not extend into a larger multiply connected portion). We see that the Sierpinski carpet's ring coincides with the carpet, while percolation clusters form multiple rings, each having a fractal dimensionality near 1.7.[11]

The idea of the squig construction is to leave carpet's D unchanged, while it is "thinned out" at random to give it the desired topological properties, by deleting in recursive manner as many bonds as necessary, but no more. There are many ways of proceeding, among which I chose the following one.

In a preliminary step, replace the usual carpet by a "dual" obtained by viewing a finite approximation to the carpet as a collection of black squares, and joining the centers of any two squares in the carpet that share a side. The approximate carpet can be viewed as the sum of eight subcarpets. Each subcarpet is linked to each of two neighbors by very many bonds, which is why the carpet itself is infinitely ramified. To achieve finite ramification, one deletes all these bonds, except one; then one does the same with the sub-subcarpets within a subcarpet etc . . . . It is proposed that this operation be called "decimation."

Next, dangling bonds are "manufactured" via a different rule of deletion, which it is proposed to call "separation." One views the already

decimated carpet as made of 8 subcarpets, plus 8 bonds linking neighboring subcarpets. With some prescribed probability s, one separates one of these bonds (with equal probabilities for the eight possibilities), and with the probability $1 - s$ one separates nothing. This rule of separation is motivated physically in Section 1.1.3. One proceeds in the same way with each part. Of course, it is best to perform both decimation and separation at each construction stage before proceeding to the next stage.

The squig cluster is defined as the outcome of this construction. By design, the overall dimensionality remains $D_S = \log 8/\log 3 \sim 1.8928$, *as desired*, and the topology is *as desired*. Now, let us dig deeper, by investigating fractal dimensionalities, then ramification properties, for several of the parts of the clusters.

1.1.2. Squig Intervals in the Squig Cluster. When the probability of separation is $s = 1$, the cluster in Section 1.1.1 reduces to a tree, that is, any two points in the squig are connected by a single path in the squig. A renormalization argument to be reported elsewhere[10] finds that the fractal dimensionality of these individual paths in a squig cluster is $D_I \sim 1.293$. For various somewhat heuristic reasons[12] the desired value is the dimensionality $4/3$ of the self-avoiding random walks. Therefore, D_I is nearly *as desired*.

When $s < 1$, however, the squig is not a tree, and in general two given points are linked by many paths. I claim that there is a natural way of choosing at random among the paths between *any* two points. One should raise s to 1, that is, one should separate each ring in the squig that had failed to be separated. This reduces the squig cluster to a tree. The natural path between any two points should then be taken along this tree; hence the path's fractal dimensionality takes the above-mentioned desirable value D_I. This tree would best be called "skeleton," but this may sound awkward since it does *not* include the backbone. Let it be called "current orbits squig tree."

1.1.3. Justification of the Notion of Current Orbit by Kirchhoff Laws. Let me now justify the procedure in 1.1.2, and thereby also help justify the definition of separation given in 1.1.1. The best is to start with the case where $s = 0$, so that initially there is no separation anywhere. Take two distinct points P' and P'' in the cluster, and assume that they lie in different eighths of the cluster. (If not, they necessarily lie in different eighths of some subcluster, and it suffices to zoom in on the largest such subcluster.) In the "renormalized" approximation, our cluster is a ring of 8 beads with 8 bonds between them, and the points P' and P'' each lie in one bead. The numbers of beads one must cross to go from P' to P'' counterclockwise (respectively, clockwise) being written as $\sigma^+ - 1$ and $\sigma^- - 1$, the quantities σ^+ and σ^- will be called + distance and − distance between P'

and P''. Now set a voltage difference between P' and P'' and follow the path of a random electron. The probability of its flowing in the + or − directions is proportional to the corresponding current intensities. Therefore, by the Kirchhoff laws, the probability of proceeding in the + (respectively, the −) direction is $\sigma^-/8$ (respectively, $\sigma^+/8$). These are precisely the probabilities implemented by my separation algorithm.

1.1.4. Decomposition of a Cluster into Rings, Glue Bonds, and Dangling Bonds. Like many physicists (e.g., Scott Kirkpatrick[6]) I favor decompositions of a graph that are intrinsic to the graph itself. In this light, a cluster decomposes into portions I shall call "rings" (see above), "glue bonds," and "dangling bonds." "Dangling bonds" can be disconnected (recursively) from the rest of the cluster by severing one link point. "Glue bonds" connect two or more rings to each other.

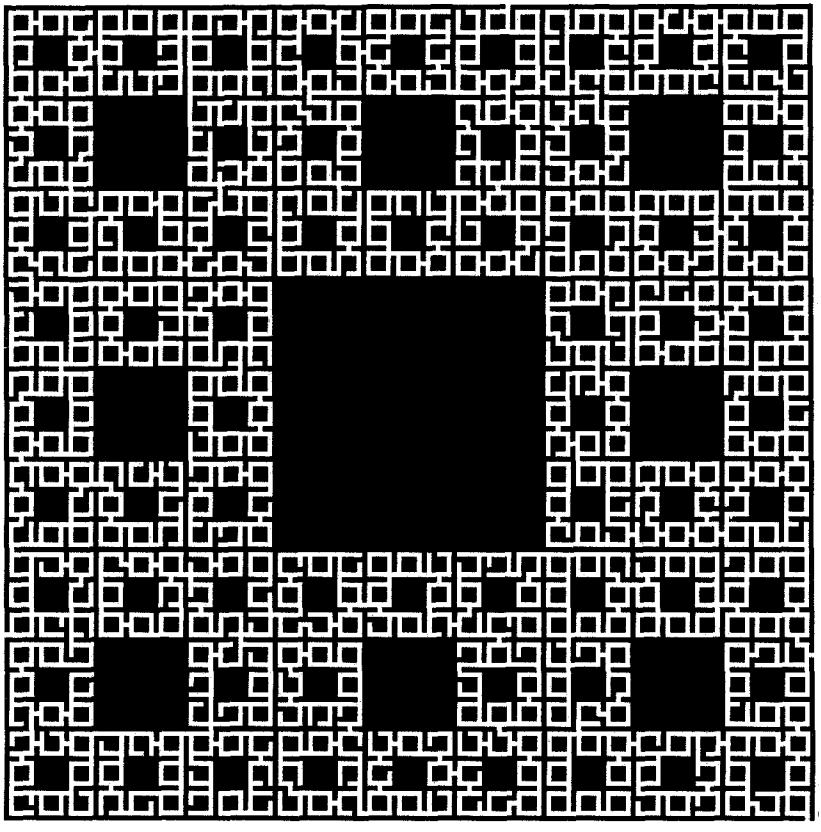

Fig. 1. Part of a squig cluster[5] matched to a two-dimensional percolation cluster.

It is easy to see that the fractal dimensionality D_R of individual rings is a continuously decreasing function $D_R(s)$ of the probability s of separation. It is clear, moreover, that for $s = 0$, one has $D_R(s) = D_S \sim 1.8928$, and that for $s = 1 - \epsilon$, one has $D_R = D_I \sim 1.2928$ (indeed, at this limit a ring is essentially reduced to two squig intervals strung together). Therefore, $D_R(s)$ can be adjusted to take any desired value D_R between $D_I \sim 1.2928$ and $D_S \sim 1.8928$. A renormalization argument[10] has determined the function $D_R(s)$ explicitly, and finds it is nearly linear. When $s \sim 0.4$, the fractal dimensionality of the ring takes the desired value ~ 1.7. See Fig. 1.

D_S is also the fractal dimensionality of all the rings taken together, of all glue bonds, and of all dangling bonds. To prove this, it suffices to apply the basic box-counting argument: at the limit of infinite interpolation each subsquare of the unit square that intersects the carpet also intersects each of these three collections of bonds. Thus, the quantity N in the expression $D = \log N / \log b$ is the same for our three sets.

Finally, the relation between linear size and number for rings is the same as reported in FGN, p. 117: it is hyperbolic with D_S as exponent.

1.1.5. Two-Point Backbone and the Set of its Glue Bonds. The rings etc. in 1.1.4 and the current orbits tree in 1.1.2 are among the multiple aspects of the broad intuitive notion of backbone. This notion digs deep into the geometric structure of a cluster, but the term "backbone" itself has been overused in contradictory ways, and should no longer be used without being specified further. Let "backbone from P' to P''" denote the portion of the cluster that carries current when a voltage difference is set between two prescribed points on plates P' and P''. Clearly, this backbone decomposes into b-rings, b-glue bonds, and b-free bonds, where b stands for backbone.

It is easy to show that this backbone's fractal dimensionality is D_R. This is obvious when P' and P'' are two points on the same cluster ring. In other cases, the proof consists in showing that the backbone's D is neither less than D_R (because the b-rings' dimensionality is D_R) nor more than D_R (because the backbone can be imbedded in a suitable collection of rings).

Now consider the set of glue bonds in a backbone. It is of central importance to the solution of the Ising problem on the cluster, and theory[13] suggests that its fractal dimensionality is 0.75. For my squig, a renormalization argument[10] finds for the fractal dimensionality of this set a function varying near-linearly from 0 to 1.2928. Selecting s to fit the backbone dimensionality, that is, $s \sim 0.4$, also reproduces the desired backbone glue dimensionality.

1.1.6. A Harmless Discrepancy Concerning Closure. Define

the "closure" of a lattice by adding all the bonds whose end points both belong to the lattice. The closure of a squig cluster reconstitutes the original Sierpinski carpet, hence includes no dangling bonds. To the contrary, actual percolation clusters at criticality include dangling bonds even after closure. This discrepancy need not be significant, however, because when two points are far from each other along the cluster, it need not matter that they are close together in Euclidean distance. Thus, while it is easy to generalize the squigs to allow the closure to be random with dangling bonds, this generalization is not explored here.

1.1.7. The Order of Ramification of the Squig Clusters and their Parts. The notion of ramification may not be familiar to all readers and it is somewhat tricky; therefore, it deserves more than a minimal discussion here. The original definition of ramification, which we shall see is still the best, proceeds by interpolation. However, physicists who work with infinitely extrapolated structures on a lattice prefer a definition based on extrapolation,[8,9] and we shall start this way. A structure's order of ramification at a site P belonging to it is the quantity $R(P)$ defined as the smallest number of bonds which one must delete in order to disconnect an arbitrarily large portion surrounding P from the rest of the structure.

Consider a squig cluster in this light. When the site is arbitrarily selected on this cluster, it is almost certain that at some early or late state of extrapolation, a square containing P will be separated from all its neighboring squares except 1. Therefore the order of ramification is $R(P) = 1$ at almost all points. In other words, $R(P) > 1$ holds for a vanishing proportion of the points. This was to be expected, because $R(P) > 1$ only holds on the infinite ring, and we know that the dimensionality of the infinite ring is D_R, like that of any individual ring, hence is strictly smaller than the dimensionality D_S of the whole infinite cluster. Recall the analogous Euclidean situation: when a point has been chosen at random in a plane in space, it is known to have a zero probability of falling on a given line in that plane.

Next, when P is chosen at random on an infinite ring, one finds that it is almost certain that an arbitrarily large portion of the cluster can be disconnected by deleting $R(P) = 2$ bonds. By constraining the site even more tightly, one can pick some points where $R = 3$, or even fewer points where $R = 4$.

An earlier fractal model of the ring that has been widely discussed (FGN, p. 132 and Ref. 9) is the Sierpinski gasket. Extrapolative ramification finds in that case that $R(P) = 3$ for almost all sites. However, $R(P) = 4$ when the very special property holds, that at *every* stage of extrapolation, P remains an apex common to two triangles. Thus, all the rings of the

gasket are attached in three points to the outside, while most of the rings of my cluster are attached in two points. The latter structure is the *more desirable one*, hence the squig is the more desirable model.

Pondering the above applications, we see that the extrapolative definition of ramification has a major defect: unless extreme care is exerted, it tends to pick out the points when $R(P)$ attains its minimum $R_{\min}$. For the gasket, this was innocuous, but for the squig cluster, it was highly misleading. It is therefore useful to go on to sketch the interpolative ramification devised in the 1920s by the mathematicians Urysohn and Menger (FGN, Chapter 14). Around each point P on an infinitely detailed fractal curve $\mathscr{C}$, they examined increasingly small balls of radius r. In each ball, they considered all the smaller neighborhoods of P that are bounded by a loop $\mathscr{J}$ (Jordan curve) that surrounds P. The smallest number of points where $\mathscr{J}$ intersects $\mathscr{C}$ is a function $R(P,r)$. The limit of that function for $r \to 0$ defines the order of ramification $R(P)$.

It is easy to see that the squig cluster includes points of order of ramification equal to 1, but also sharply decreasing numbers of points where the order of ramification is equal to 2, 3, or 4. For example, 3 is at almost all of the 7 (or 8) points that joint the eights of the largest ring in the cluster, the 7 (or 8) points that joint the eights of each eight, etc.

1.1.8. Clusters Above or Below Criticality. One can proceed as in FGN, p. 129. Just above criticality, there is a length scale ξ such that percolation clusters much larger than ξ are homogeneous and those smaller than ξ are indistinguishable from critical clusters. Simulations[14] reveal that the transition at ξ is very sharp. Therefore, it may suffice to take a grid of square cells of side ξ, to position statistically independent squig clusters in each cell, and to activate a randomly selected bond between neighboring clusters. This process creates an infinite cluster, but the size distribution of clusters of side $< \xi$ is unchanged.

To simulate the situation just below criticality, the only change is that neighboring clusters of size ξ in the above square grid are left separated. A smoother transition is achieved by making the probability of separation s into a function of size: constant over small sizes then drifting towards 1 for larger sizes. Other rules of change of s with size achieve other cluster size distribution.

1.2. General Principles Underlying the Notion of Side Tied Squig

In Section 1.1, the general principles of construction of squigs enter together, in one specified combination. Now they will be introduced one by one, to demonstrate the versatility achievable by suitable selections from a

single very large "menu." It will be seen that a squig's topology may be either left to be determined by chance, or prescribed in advance. In the latter case, the squig may be an interval or a straight line, but it may also be a tree or a collection of rings with or without dangling bonds. If it is a curve, it may be either finitely or infinitely ramified, but the stress will be on the former. (It may also be a surface, but this generalization is not tackled here.)

Recursivity is built into the squig construction by starting with a triangular or square lattice whose cells are then collected in super cells of size b^k, where b (the base) and k are integers. (One can also use more general lattices that "pertile" in the sense of FGN, p. 46, but none will be mentioned here.) Recursivity allows renormalization group arguments to be carried out. I believe them to be mathematically rigorous, and this belief has been explicitly confirmed in several basic cases, in the papers by J. Peyrière.[15] These papers do not use the term "renormalization." However, the basic idea is there. In fact, it has long been known to mathematicians. To take an example, renormalization underlies the search for central limit theorems of probability theory, where the fixed points have been identified by Cauchy, and the domains of attraction (physicists' "universality") by P. Chebyshev and Paul Lévy.

While earlier papers[5,15] and FGN Chapter 14 proceed by interpolation, here we proceed by extrapolation, and "fractal" will denote a finite approximation to a mathematical fractal.

Squigs are *not* drawn on the original triangular or quadratic lattice, but on a hexagonal or square lattice of "potential bonds," obtained by linking the centers of "neighboring cells" in the original lattice, that is, of cells that share a side. Some bonds are then deleted at random, in recursive fashion, using one of two distinct processes: "decimation" and "separation." The remaining bonds are called "activated." The construction is tractable because the processes of decimation and separation will be statistically independent.

1.2.1. Pure Decimation on a Triangular Lattice of Base $b = 2$. A Plane-Filling Squig Tree. The first construction stage takes b^2 triangular lattice cells $C(0)$ that fit into a twice larger triangle $C(1)$, and activates all the b^2 bonds between neighboring $C(0)$'s. The result is a small Y-shaped tree. The second construction stage takes b^2 copies of $C(1)$ that fit together in a triangle $C(2)$ whose linear size is b times larger. The boundary between any two neighboring $C(1)$ within $C(2)$ is crossed by b potential bonds; one of them is activated and the remaining $b - 1$ are decimated. The kth stage takes b^2 statistically independent replicas of $C(k-1)$ to make a b^2 larger triangle $C(k)$. The boundary between any two neighboring $C(k-1)$ is

crossed by b^{k-1} potential bonds; one of them is activated, and the other $b^{k-1} - 1$ are decimated.

The final outcome is a connected fractal, a "squig tree," that can be said to fill the whole plane, hence is of dimensionality $D = 2$.

Digression. To be more precise, let a squig tree be drawn by interpolation, so that rigorous Hausdorff dimensional arguments can be carried out. One finds that one deals with the whole plane, minus a denumerable collection of intervals, whose removal chops up the plane into a tree. Thus, one deals with a set of dimensionality $D = 2$, *minus* a set of dimensionality $D = 1$. One is reminded of the structure of the irrational numbers on [0, 1], which are the difference between an interval ($D = 1$) and a denumerable collection of rational points ($D = 0$). One is also reminded of the lengthening list of fractals whose fine structure is described intrinsically as the sum or difference of several partial fractals, hence involves several fractal "eigendimensionalities" (see Section 3.3 below, FGN, p. 151 and Ref. 16). "Closing" all the bonds that are decimated in the present construction yields the addend that has the largest eigendimensionality, which in the present case is the whole plane.

1.2.2. Pure Decimation on a Square Lattice of Base $b = 2$. A Plane-Filling Squig Ring. The first construction stage takes b^2 square lattice cells $C(0)$ that form together a b times larger square $C(1)$, and activates the b^2 bonds between neighboring $C(0)$'s. The result is a small square ring. The words that describe the following stages are exactly as in 1.2.1.

The final outcome, again, is connected and fills the whole plane, but is otherwise very different from the final outcome in 1.2.1. It is a connected collection of fractal rings, that is, of rings upon rings, without dangling bonds.

1.2.3. Pure Decimation on a Triangular Lattice of Base $b = 3$. A Plane-Filling Squig Cluster. All the words that describe the construction are the same as in 1.2.1.

The final outcome, again, is connected and fills the whole plane, but is otherwise very different from the result in 1.2.1. It is made of rings, glue bonds, and dangling bonds.

1.2.4. Discussion of Generalizations, in the Plane and Beyond. To extend pure decimation to higher values of b and to Euclidean spaces of dimensionality > 2 is easy. The next important operation of squigs shown is "separation." It is a form of deletion that differs from decimation. Starting with a plane-filling structure obtained by pure decimation, separation can have either of several effects. In 1.2.5, prescribed separation serves

to chop up a single plane-filling infinite squig into a collection of finite squigs. In 1.2.6, randomly positioned separation serves to open all the rings of the fractal squigs drawn in 1.2.1, 1.2.2, and 1.2.3, thus yielding either a plane-filling tree or a collection of bounded trees. In 1.2.7, separation is randomly decided and randomly positioned. Thus, it serves to obtain either a plane-filling mixed cluster, or a collection of mixed clusters. The latter are the squig clusters proposed in 1.1 as models for percolation.

1.2.5. Example of Decimation and Prescribed Separation on a Square Lattice of Base $b = 3$. Sierpinski-Carpet-Filling Squig Rings. In this case, the construction stage takes 9 square lattice cells $C(0)$ that form together a 3 times larger square. These squares are linked by 12 potential bonds. All the 4 bonds that involve the central square $C(0)$ are "separated," and the 8 other bonds are activated. The result, to be denoted by $C(1)$, is a square, each side of which is made of two colinear bonds. The second construction stage takes 9 copies of $C(1)$ that fit together in a 3 times larger square. There are 36 bonds. One begins by separating all the 12 bonds that involve the middle square $C(1)$. This leaves 3 potential bonds between any two neighboring $C(1)$ that are not separated. One of these bonds is activated and 2 are decimated. The kth stage takes 9 statistically independent replicas of $C(k-1)$ that fit into a 3 times larger square. The middle replica is separated from the rest. This leaves 3^{k-1} potential bonds between any two neighboring $C(k-1)$ that are not separated: one of these bonds is activated and the other $3^{k-1}-1$ are decimated.

The final outcome is an infinite collection of nested squig ring structures. One can define the "closure" of this fractal as the lattice obtained by completing all the bonds that join two points in the structure. This closure is a finite lattice approximation to a Sierpinski carpet. The fractal dimensionality of each structure is the familiar $D = \log 8/\log 3 \sim 1.89$. The number-size relation is also familiar: as the linear size of a structure is multiplied by 3, the number is multiplied by $8 = 3^D$.

Digression. The digression in 1.2.1 is also relevant here. The above structure is a Sierpinski carpet of fractal dimensionality $D = 1.89$, minus the decimated set of dimensionality $D = 1$ that is used to chop up the carpet.

1.2.6. Systematic Randomly Positioned Separations, Superimposed upon the Decimations in 1.2.2, 1.2.3, and 1.2.5. Squig Trees. Resuming the construction in 1.2.2, let us modify it by "separating" each ring in one place. In the first stage, one of the four bonds between neighboring $C(0)$'s is chosen at random (equal probabilities) and is separated, while the remaining three bonds are activated. Similarly one of the four nondecimated bonds between neighboring $C(k-1)$'s is chosen at

random and separated while the remaining three are activated. This systematic random separation creates an infinite squig tree.

The same process can be used to modify the construction in 1.2.3, except that there are six nondecimated bonds between neighboring $C(k-1)$'s. This systematic random separation again creates an infinite squig tree.

The same process can also be used in 1.2.5, except that there are eight nondecimated bonds between the nonseparated neighboring $C(k-1)$'s. This systematic random separation creates an infinite collection of bounded random squig trees.

1.2.7. Randomly Decided and Randomly Positioned Separations, Superimposed upon the Decimations in 1.2.2, 1.2.3, and 1.2.5. Squig Clusters. The next step is obvious: instead of performing randomly positioned separations in *all cases*, as done in 1.2.6, or in *no case*, as done in 1.2.2, 1.2.3, and 1.2.5, let us perform separation in *some cases*, chosen at random, independently of each other. This requires introducing a freely adjustable parameter s: the probability of performing a separation.

The final outcome combines rings, glue bonds, and dangling bonds. When the point of departure is as in 1.2.5, the point of arrival is the squig cluster in 1.1.

1.2.8. Squig Intervals. A squig interval is a path that joins any two points on a squig tree. All other squig fractal curves, e.g., trees and squig clusters, can be viewed as unions of squig intervals. The striking finding is that, in all the above listed examples, the squig's intervals' fractal dimensionality D_I is very close to $4/3$.

The construction in 1.2.1 yields squig intervals with $D_I = \log 2.5/\log 2 = 1.3219$. The reason is that the number N of $C(k-1)$ that belong to a $C(k)$ and contain a squig interval satisfies $\langle N \rangle = 2.5$. From a renormalization group argument,[5] I inferred that $D = \log\langle N \rangle/\log 2$, which Peyrière[15] shows to be an exact result.

The construction in 1.2.2 yields squig intervals with $\langle N \rangle = 13/3$, hence $D_I = \log\langle N \rangle/\log 3 = \log(13/3)/\log 3 = 1.3347$.

In the construction in 1.2.3 the formula for D is more complicated. One finds that 2^D is the leading eigenvalue of a "transfer matrix"[16] with the lines $5/4$ and $3/4$ (top) and $6/4$ and $6/4$ (bottom). This leading eigenvalue is 2.4430, hence $D_I = 1.2886$. The other eigenvalue is 0.030, and it contributes a slight corrective term.

The construction in 1.1 (i.e., the last stage in 1.2.7) had yielded squig intervals with $D_I = \log 4.1266/\log 3 \sim 1.2928$.

Remark A. The value $D = 4/3$ is of course also known to occur in self-avoiding random walks, SARW. This is likely to indicate that the

SARW and squig constructions both underscore some vital but not yet identified mathematical property of the plane.

Remark B. In FGN, p. 225, a direct construction of a squig interval was motivated by my unhappiness with Wiener's construction of the Brownian motion in the plane and with von Koch's construction of the snowflake curve. Both constructions involve recursive interpolation, but approach different limit points at very different rates. Indeed, they begin with *exactly* known positions $P(0)$ and $P(1)$ and *totally* unspecified intermediate positions. Then, they pin down *exactly* the intermediate positions for dyadic values of t, following the order $P(2^{-1})$, $P(2^{-2})$, $P(3.2^{-2})$, etc. . . . . For the purpose of the physicist's renormalization, as well as of the mathematician's search for elegance, it would be far preferable to define a curve as the limit of strips whose width is uniform at each stage and narrows down to 0. This goal is achieved by the squigs.

1.2.9. Infinitely Ramified Squigs Constructed Recursively. The idea is of course to achieve infinite ramification, but to a degree smaller than in the original uncut plane. To give an example of how this can be done recursively, let us start with the construction in 1.2.5. In the second stage, between any two neighboring $C(1)$ that are not separated there are three potential bonds. In 1.2.5, we activate $N = 1$ of these bonds, but the number of activated bonds can be chosen differently. It may be either prescribed and > 1, or random with $\langle N \rangle > 1$. When N is random, the values to be used at different locations are to be statistically independent. Once N is chosen, activate a combination of N bonds selected at random among all the possible combinations. In the kth stage, the 3^k potential bonds between neighboring $C(k-1)$ are activated according to a hierarchical screening scheme. The first screening separates the 3^k bonds into three groups of three, and decimates all but N of them. Then each nondecimated group of 3^{k-2} bonds is screened separately, and all but N are decimated. In this way, each nondecimated bond in 1.2.5 is replaced (asymptotically) by a Cantor dust of dimensionality $\log\langle N \rangle / \log b$.

2. PART TWO: LATENT FRACTAL DIMENSIONALITY, NOISES, AND DIFFUSIONS

2.1. Algebra with Fractal and Latent Fractal Dimensionalities

A striking characteristic of the notion of fractal dimensionality D is that, under suitable circumstances, a very precise geometric meaning attaches to the *sum* of the dimensionalities of two distinct sets, their *product*,

Table I. Dimensionalities of Fractal Sets Generated Either by $B_H(t)$ in $\mathbb{R}^D$, or by $L_\alpha(f)$ in $\mathbb{R}^D$, or by an Independent Random Walk on a Fractal of Dimensionality D

Specification of a space, or of a fractal set or description of a criticality criterion	Fractal dimensionality (Given in latent value)
1. Maximal range; this may be the space $\mathbb{R}^D$	D
2. Trail (irrespective of sequence)	$1/H = \alpha$
3. Upper critical D for space-filling by the trail	$D = 1/H = \alpha$; $D/\alpha = DH = 1$
4. Zeroset on the time axis for $D = 1$	$1 - H = 1 - 1/\alpha$
5. Zeroset on the time axis for arbitrary D	$1 - DH = 1 - D/\alpha$
6. Upper critical D for infinite recurrence at all points	$D = 1/H = \alpha$; $D/\alpha = DH = 1$
7. Criterion common to space-filling and to recurrence	$D/\alpha = DH < 1$
8. N-tuple points in the trail	$D - N(D - 1/H) = D - N(D - \alpha)$
9. Upper critical D for N-times recurrence at some points	$D = N/H(N - 1)$; $D/\alpha = DH = N/(N - 1)$
10. Upper critical N for N-times recurrence	$N = 1/(1 - 1/DH) = 1/(1 - \alpha/D)$
11. Example: upper critical D for 2-times recurrence at some points	$D = 2/H = 2\alpha$; $D/\alpha = DH = D + 1 - DH$
12. Graph, in the affine space $\mathbb{R}^D \times$ (time), of a motion in $\mathbb{R}^D$	
13. Motion in $\mathbb{R}^D$, when time is restricted to the instants when $F < D$ of the coordinates vanish: trail	$(1/H)(1 - FH) = 1/H - F = \alpha - F$
14. Same time instants as on line 13: graph	$D + 1 - F - EH$
15. Points for the trail where the first and second coordinates are equal	$1/H - 1 = \alpha - 1$
16. Same condition as on line 15: Points on the graph	$D - DH$
17. Formal dimensionality of the (formally Euclidean) wave-number space	$2DH$

their *quotient*, their *difference*, and their *cosum*. The cosum is defined as the sum of the codimensionalities, which are defined in the Euclidean space of dimensionality E by the formula $C = E - D$. A few examples of each of these operations are classical and are discussed in FGN, and several new examples have emerged recently. A selection is included in Table I. Cosums and products are also discussed separately in 2.1.2 and 2.1.3.

Under certain circumstances, these various operations combine several dimensionalities by very simple rules, but under other circumstances the rules are complicated. This is very irritating. However, a fact has been apparent from the outset—for example, in the work of Hutchinson[17]—and is becoming increasingly clear and important. The fact is that all these complications disappear if a "latent fractal dimensionality" is attached suitably to certain mechanisms that generate fractal sets. In the case of fractals imbedded in the Euclidean space $\mathbb{R}^E$, the latent value is allowed to be either $> E$ or < 0, and bears the following relation to the "actual" value.

Whenever the latent fractal dimensionality is $> E$, the actual fractal dimensionality is E.

Whenever the latent fractal dimensional is < 0, the actual fractal dimensionality is 0.

2.1.1. Loss of Equivalence Between the Fractal and the Hausdorff Dimensionalities. One recalls that an important role in the development of fractal geometry has been played by the Hausdorff dimensionality D_H, a quantity that necessarily satisfies $0 \leqslant D_H \leqslant E$. Now, by the very fact of defining the latent dimensionality, we necessarily *disassociate* the notions of fractal and of Hausdorff dimensionality. Furthermore, when a set's actual fractal dimensionality is 0, its topological dimensionality D_T can be either 0 (when the set is a dust) or -1 (when it is empty). Thus the inequality $D_H \geqslant D_T$ fails to have a counterpart when D_H is replaced by the latent fractal dimensionality.

Other reasons for disassociating the fractal and the Hausdorff dimensionalities are encountered and discussed in the sequel, and are brought together in Part 4. The past association between the two should not be mourned, because there are extremely few cases of concrete interest where the Hausdorff dimensionality can be computed explicitly and rigorously. *Fractal* dimensionality should remain a "generic" notion, the definitions by Hausdorff *et al.* being specific implementations.

2.1.2. Intersections, and the Addition Rule for Codimensionalities. By Introducing "Latent" Fractal Dimensionalities That Can Satisfy $D < 0$, One Generalizes the Scope of the Simple Part of This Rule, and One Implements an Intuitive Notion of "Degree of Emptiness" of a Set. Consider, in the Euclidean space $\mathbb{R}^E$, the intersection of

two sets of respective dimensionalities D' and D''. For the Hausdorff dimensionality, the rule of thumb is that the intersection's codimensionality $E - D$ is the sum of the codimensionalities $E - D'$ and $E - D''$; but we have *the following major exception*: when this rule yields a negative D, the actual D is 0.

This exception is an irritating complication, and it hides a feature that is in fact worth underlining. As background, compare the intersections of two planes, a plane and a line, and two lines, all in the space $\mathbb{R}^4$. All three intersections are well known to be empty in general, and this is confirmed by the rule relative to the Hausdorff dimensionality. But intuition also tells us that the intersection of two lines is "emptier" than the intersection of a plane and a line, or the intersection of two planes. This loose notion fails to be expressed by the value of the Hausdorff dimensionality. On the other hand, the seemingly "thoughtless" addition of codimensionalities yields -2, -1, and 0 for the dimensionalities of the three intersections that we consider. Therefore, this addition is not thoughtless at all, and its result is a very useful measure of the relationship between two nonintersecting sets. Of course, this result is *no longer* a Hausdorff dimensionality, but it can be called a "latent" value of the fractal dimensionality. When the latent value is < 0, the actual value is 0.

2.1.3. Fractal Subordination, and the Rule of Multiplication of Dimensionalities. By Introducing Latent Fractal Dimensionalities Whose Values in $\mathbb{R}^D$ May Exceed D, One Implements an Intuitive Notion of Degree of Coverage of the Plane by a Set. The process of subordination of fractals is a very important one, discussed in detail in FGN, Chapter 32. The basic example starts with the trail of Brownian motion, whose dimensionality will be written as $\alpha = 2$, and "lights" it very intermittently: only when time falls within a fractal dust ("Cantor-like set") of dimensionality D_S. The set of points that are lighted is known to have the dimensionality αD_S. A generalization of Brownian motion is fractional Brownian motion $B_H(t)$, which is investigated in FGN, Chapter 27 and sketched below in Section 2.3.1. This motion's exponent H lies between 0 and 1, and there is a lower critical dimensionality equal to $\alpha = 1/H$. When $D > \alpha$, the dimensionality of the fractional Brownian trail is α, and the dimensionality of the instants when the trail is lighted is, as above, the product of dimensionalities $\alpha \cdot D_S$. However, suppose that $D < \alpha < D/D_S$; then the dimensionality of the instants when the trail is lighted is again αD_S, but this quantity is no longer obtained as a product of dimensionalities. To allow it to continue to be called a product of dimensionalities, I propose that, irrespective of the sign of $D - \alpha$, the trail can be said to have a "latent fractal dimensionality" equal to α. This value can lie above D. When the latent value is $\alpha > D$, the actual value is D.

2.2. On Several Different Kinds of Scaling ("$1/f$") Noise: Fractional Gaussian Noise Versus White Noise on a Fractal

Many noises have a scaling observed spectral density, that is, a density that is proportional to $1/f^B$. Therefore, they are called "$1/f$ noises." While the physical mechanisms behind $1/f$ noises remain obscure, several papers I wrote in the 1960s[18,19] have sorted out their geometry. In particular, I showed that scaling may be traceable to either of several very different reasons, the following three extreme cases being the most important.

2.2.1. Gaussian Noises. They are familiar to physicists, and are best viewed as the derivatives of the fractional Brownian functions $B_H(t)$ discussed in 2.3.1. They are "on" all the time.

2.2.2. "Sporadic," "Absolutely Intermittent," Noises. They are "off" (equal to a constant—usually to zero) almost all the time, and "on" (varying or nonzero) only when time falls within some fractal dust ("Cantor set"). The simplest examples of sporadic noises with $B > 1$ are provided by the error sequences first investigated by Berger and Mandelbrot in 1962 (FGN, Chapter 8) and described in FGN, Chapter 8, and by related examples in Refs. 18 and 19. One of these examples[19] was contrived on purpose to appear to be Gaussian in one form of analysis and extremely non-Gaussian in another.

2.2.3. "Relatively Intermittent" Noises. They are "on" all the time, but most of the time their value is imperceptibly small. They expend nearly all their energy over instants that form a fractal dust. They are described in FGN, pp. 375–381 and below in 3.2. Records of dissipation in intermittent turbulence are the most important example.

The contrast between a highly anomalous ("nonwhite") noise in ordinary Euclidean time (2.2.1) and an essentially nonanomalous noise in fractal time (2.2.2) is a special case of, and is entirely analogous to, the contrast between the diffusions examined in greater detail in 2.3.

2.2.4. Significance of the Energy in Very Low Frequencies. The $1/f^B$ spectrum means that very low frequency components have very high energy. This manifests itself very differently in the extreme forms of scaling noise. In 2.2.1, this energy is due to something that is *present*: a low rumble. In 2.2.2, low frequencies are traceable to something that is *absent*: to the fact that there is no energy variation at all during the long "off" or constant periods between bursts of high-pitched noise. An arbitrarily chosen sample usually will fall in a long gap between noise bursts. Therefore, the theory of this process must concentrate on samples that are "conditioned" to be nonconstant. As conditioned samples of increasing duration

T are considered, increasingly long silent periods are observed, and as a result the total energy in the high-pitched bursts increases less than linearly. As the frequency range over which the spectrum takes the form $1/f^B$ increases towards 0, the average distribution of energy moves along the spectrum towards the low frequencies. The $1/f^B$ spectral density seems to threaten a low-frequency divergence of "infrared catastrophe," but this threat never materializes, because the spectral density is $F(T)/f^B$, with a prefactor F that decreases as T increases.

2.3. Two Well-Understood Forms of Anomalous Diffusion: Fractional Brownian Motion, and Lévy Stable Motion

Sections 2.3 and 2.4 move on from noises to anomalous ("non-Fickian") diffusions. Interest has recently[20,21,22,23,24] focused on problems related to an "ant in the labyrinth," which takes independent random steps while restricted to a specified substrate, namely, to a fractal that has been previously constructed (by either a nonrandom or a random process).

The study of this ant would have benefited by being placed against the background of two very different anomalous diffusions that are thoroughly understood and play a central role in FGN: these diffusions are fractional Brownian motion and Lévy stable motion. This and the next sections propose to provide such a background, and to extend the known results a bit. The main exhibit is Table I, and its point is that the same basic fractal formalism applies to the three anomalous diffusions mentioned in this paragraph. One should expect this formalism to apply even more widely.

2.3.1. The Fractional Brownian Motion $B_H(t)$. This is a continuous function of time, with values in $\mathbb{R}^D$, whose increments are Gaussian, have zero mean, and satisfy $\langle B_H(t) - B_H(0)^2 \rangle = |t|^{2H}$, with an exponent satisfying $0 < H < 1$. The exponent $H = 1/2$ corresponds to ordinary Brownian motion, whose increments are uncorrelated—hence independent. Exponents satisfying $1/2 < H < 1$ correspond to *persistent diffusion*, whose increments are *positively* correlated on all time scales. And exponents satisfying $0 < H < 1/2$ correspond to *antipersistent diffusion*, whose increments are *negatively* correlated on all time scales. The "next step" of $B_H(t)$ is highly correlated with all the "past steps," but is allowed to move anywhere in a Euclidean space $\mathbb{R}^E$. As it moves on, it generates an underlying fractal curve. The function $B_H(t)$ on the line is best known to many via the vertical sections of my Brown model of landscape, FGN, Chapter 28, and it also enters in numerous other models and theories described in FGN.

$B_H(t)$ has been thoroughly studied by mathematicians, as sketched (with many references) in FGN, Chapter 39, and Table I also incorporates many facts about it that had not been worth recording until now.

2.3.2. The Lévy Stable Motion $L_\alpha(t)$. This is a discontinuous function time, with statistically independent jumps that are allowed to move anywhere in $\mathbb{R}^D$ and follow the probability distribution $\Pr(U > u) = u^{-\alpha}$. As $L_\alpha(t)$ moves, it generates an underlying fractal dust, best known to many through my crude "Seeding of the Universe" model of galaxy distribution, FGN, Chapter 32. Describing it in detail here would probably look repetitive, but Table I includes the properties of $L_\alpha(t)$.

2.3.3. Notation. The letter H (which I picked in 1965; it is the initial of H. E. Hurst) is now deeply rooted in the literature on $B_H(t)$, and the letter α (which Lévy picked in the 1920s) is standard in the many papers on $L_\alpha(t)$. Therefore, both notations are best left unchanged. To help Table I underline the points common to the two processes, it is useful to *define* α as $1/H$ for $B_H(t)$ and to define H as $= 1/\alpha$ for $L_\alpha(t)$. It is of course my hope that these letters will also be adopted in the study of the ant in the labyrinth and of other independent random work on prescribed fractals, where the counterpart of $\alpha = 1/H$ has (confusingly!) been denoted by $2 + \theta$, $2 + \bar{\delta}$, $1/\nu_{\text{rw}}$, or D.

2.4. Comments on Table I

Several distinct fractal sets enter into the study of $B_H(t)$ and of $L_\alpha(t)$. Table I brings their fractal dimensionalities together and also lists the corresponding criticality criteria. The result is a well-defined mathematical framework in which one ought to place the study of random walks on percolation clusters or on other specified fractal nets. The fractal dimensionalities are given in the "latent" form introduced in Section 2.1. They are functions of the scaling exponents H and α linked by $\alpha H = 1$. Suppose that a fractal of latent dimensionality α overfills the space $\mathbb{R}^D$ or some prescribed fractal having the fractal dimensionality D, in the sense that $\alpha > D$; then the actual fractal dimensionality is D. When the latent fractal dimensionality falls below 0, the actual fractal dimensionality is 0.

Observe that different lines on this table obtain from other lines as sums, differences, products, and quotients. The table includes both general rules and special cases thereof.

Range of α or H for B_H and L_α. The function $B_H(t)$ requires $0 < H < 1$, hence $1 < \alpha < \infty$. And $L_\alpha(t)$ requires $0 < \alpha < 2$, hence $1/2 < H < \infty$. Overall, the whole range of positive values of α and H is covered, the subrange $[1/2, 1]$ being covered in either of two ways.

Range of D/α or DH. One encounters "recurrent" cases when $DH < 1$ as well as "non-recurrent" cases when $DH > 1$.

The "Fracton" Terminology. The Alexander–Orbach Conjecture. In the "ant in the labyrinth" problem, the counterpart of the quantity on line 17 has been called "fracton" or "spectral" dimensionality.[21] I dislike both

terms, but terminology should not matter. The more important point is that the basic probabilistic notion in the study of diffusions is not $2HD$, but the codimensionality HD in line 5. Thus, the best statement of the conjecture of Alexander and Orbach[21] is that in every $\mathbb{R}^E$, a random walk on a percolation cluster recurs to the origin at instants that have the dimensionality $1/3$, independently of E.

Warning Concerning Line 12. In the study of isotropic fractals in Euclidean spaces, dimensionalities enter as exponents in expressions of the form $M(R) \propto R^D$, meaning "mass in a sphere or interval of linear size $R \propto R^D$." However, the space $\mathbb{R}^D \times$ (time axis) is not a Euclidean but an affine space, in which rotations are meaningless and distance along the time axis cannot be compared with the distances along the space axes. In such a space, a sphere cannot be defined, R is meaningless, and D *cannot* enter in as exponent.

Comparison of Lines 4 *and* 5. For functions in $\mathbb{R}^D$, line 5 follows immediately from line 4, because $B_H(t) = 0$ when all the D coordinates of $B_H(t)$ vanish simultaneously, i.e., in the intersection of the zerosets of its coordinates. By the rule of thumb in Section 2.1.2, the codimensionality of the intersection is the sum of the codimensionality of the intersecting sets.

Comparison of Lines 3 *and* 6. For functions in $\mathbb{R}^D$, the existence of a common criterion of criticality is simply due to the fact that $\alpha - D$ and $1 - D/\alpha$ have the same sign. When D is below α, the inequality $\alpha > D$ implies that the trail overfills $\mathbb{R}^D$, and the inequality $1 - D/\alpha > 0$ implies that the trail recurs to the origin. When D is above α, the inequality $\alpha < D$ implies that the trail fails to fill $\mathbb{R}^D$, and the inequality $1 - D/\alpha < 0$ implies that the trail fails to recur to the origin. This association between overfilling and recurrence is intuitively extremely reasonable.

The Criterion of Criticality $D/\alpha = DH = 1$. It appears to be very tight. For example, ordinary Brownian motion in the plane satisfies $H = 1/2$ and $D = 2$, hence it is critical. This motion is everywhere dense, and one can say that it fills the plane, but only barely (see FGN, Plate 243). And it fails to recur, but fails barely, since a random walk does recur in a plane lattice as opposed to the whole plane. On the other hand, ordinary Brownian motion on the line $H = 1/2$ and $D = 1$ grossly overfills the line, and is well known to recur on a zeroset of dimensionality $1/2 = 1 - 1/2$. Ordinary Brownian motion does *not* fill $\mathbb{R}^D$ with $D > 2$, and does *not* recur.

Those physicists who do not hesitate to manipulate formal "Euclidean" spaces of fractional dimensionality will argue that $B_H(t)$ is critical in the space $\mathbb{R}^{1/H} = \mathbb{R}^{\alpha}$.

Comment on Line 10. *Facts and Hypotheses Concerning the Order of Ramification.* The case $D = 2$ and $H = 0.5$ yields Brownian motion in the plane. I have previously conjectured (FGN, p. 243) that its trails have an

infinite order of ramification, in fact, that they are universal (plane) curves in the sense of Sierpinski. This was proven by S. Kakutani and Tongling (unpublished).

I now conjecture that the same is true in all the critical cases $DH = 1$. Further, I conjecture that, when $DH > 1$, the order of ramification of the trail is the integer part of $1/(1 - 1/DH)$.

Comment on Line 11. *The Criterion of Criticality* $DH = 2$. When $DH > 2$, the graph $B_H(t)$ has no multiple point of any order.

Comment on Line 13. As a special example, consider two independent *fBm*, namely $B'_L(t)$ and $B''_L(t)$, and examine $B'_L(t)$ at those instants of time when $B''_L(t)$ recurs to its original position $B''_L(0) = 0$. This set is obtained by subordination, hence its actual fractal dimensionality is $\min\{D, L\min(0, 1 - D/L)\} = \min\{D, \min(0, L - D)\}$. Three cases must be distinguished: $L > 2D$, $D < L < 2D$ and $L < D$, yielding, respectively, the fractal dimensionalities D (space filling subordinate), $L - D$, and 0 (empty subordinator, hence empty subordinate). The latent fractal dimensionality is of course to be defined as $L - D$.

Comparison of Lines 1 *and* 12. *Effects of Projection.* By inspection, we have the inequality $\min(D, 1/H) \leqslant D + 1 - DH$. Attainment of criticality, $DH = 1$, expresses that equality prevails, meaning that projecting the graph on $\mathbb{R}^D$ in order to obtain the trail leaves the fractal dimensionality unchanged. Otherwise, this projection decreases the dimensionality. It is inconceivable that dimensionality should *increase* by projection, therefore the inequality $\min(D, 1/H) \leqslant D + 1 - DH$ could not have failed to hold.

The rule of thumb valid in Euclidean spaces is that a set of dimensionality D' in $\mathbb{R}^D$ projects upon $\mathbb{R}^F$ (with $F < D$) on a set of dimensionality $\min(F, D')$. Except at criticality, it is seen that this rule of thumb *fails* to generalize from Euclidean to affine spaces.

Comparison of Line 13 *for* D *and for* $F < D$. This comparison is equivalent to projection from $\mathbb{R}^D$ onto $\mathbb{R}^F$. The dimensionality on line changes by $(D - F)(1 - H)$. Again the customary rule of thumb *fails* to generalize from Euclidean to affine spaces.

3. PART THREE: FRACTALS COME IN MANY SHAPES, INCLUDING SETS, MEASURES AND GRAPHS. EACH FRACTAL SET HAS A UNIQUE FRACTAL DIMENSIONALITY, BUT FRACTAL MEASURES HAVE AN INFINITE NUMBER OF FRACTAL DIMENSIONALITIES

The fractals involved in Parts 1 and 2, and in the bulk of FGN, are point sets, more precisely, compact sets in Euclidean space. The notion of fractal, however, extends beyond compact sets. This is one of many reasons

why attempts to define fractals too soon in too precise a way have turned out to be ill-inspired (see Section 4). Special interest attaches to fractals that are not point sets but graphs or measures.

Pure mathematicians like A. S. Besicovitch had extended the scope of Hausdorff dimensionality to what I have since proposed to call fractal measures, but this extension was never claimed explicitly. My work on turbulence (see 3.2) has long made heavy use of fractal measures, and there is a chapter on them in Ref. 1; however, I had not foreseen their rapid emergence at center stage, and FGN sketches them all together all too briefly in a subchapter of Chapter 39, calling them "nonlacunar fractals." I immediately ceased to like this term; therefore, jumping into hasty terminology—a tendency I have been known to criticize in other writers—is a failing to which I am not totally immune.

Fractal graphs come from a different tradition, that of physics of systems in which interactions run between sites that are neighbors on a graph and may not run between neighbors in an imbedding Euclidean space. References will be given in Section 3.4.

3.1. Basic Rule: When a Point Set of Interest to Physics is Fully Specified, There is a Single Well-Defined Fractal Dimensionality D

Yesterday, when the loose notion of fractal dimensionality was part of pure mathematics, there was much fun in devising alternative definitions, and either in proving their equivalence or inventing sets for which these variants yield different values. These definitions have been compared in Refs. 1, 2 (Chapter 12), and 3 (FGN, Chapter 39). There, as in the rest of my work, it is either stated or implied that the cases when different mathematical definitions of fractal dimensionality truly disagree are, at present, of no interest in physics. Nevertheless, other authors' recent reexaminations of the same sets of definitions (and of a few I had thought need not be examined) repeatedly seem to yield conflicting dimensionalities. Some authors to be discussed below have reacted by restricting the term "fractal" to one of these definitions. This suggestion is not only capricious, but grossly misleading because it suggests differences in kind that in fact do not exist.

To many people, this discrepancy has become irritating, but I propose to show it is not serious. Typically, it reveals that it is now necessary to go beyond strictly self-similar fractal sets (which have become well understood, hence are viewed as simple). Increasingly diverse situations that intrinsically involve a multiplicity of point sets are often encountered in the study of fractal measures and fractal graphs. Thus, the basic rule in the title

of 3.1 implies that cases of indeterminacy of fractal dimensionality will vanish if this term is never applied to phenomena ("percolation," "turbulence, . . . ") only to fully specified sets.

3.1. First Qualification to the Basic Rule: Not Every Anomalous Dimensionality is a Fractal Dimensionality. Typical of scientific terminology, the term "dimensionality" tends to be used loosely. Whenever an expression coincides with dimensionality in the Euclidean case and generalizes formally to the fractal case, someone seems tempted to use it to define a fresh "generalized" or "anomalous" dimensionality. To police those who yield to this temptation would be impossible, yet one must observe that in many instances use of the term "generalized dimensionality" reveals a careless or unfinished investigation. Dimensions that multiply without necessity (to paraphrase William of Ockham) *must* not be encouraged. Anyhow, as of today, the well-defined dimensionalities whose values differ from the fractal dimensionality for many point sets appear to be irrelevant to physics.

First Example: *The Fourier Exponent.* An exponent D_F, often called Fourier dimensionality (FGN, p. 360), enters in some fine points at the interface of arithmetic and Fourier theory. For the so-called Salem sets, $D_F \equiv D$. For non-Salem sets, $D_F < D$. However, this inequality expresses that a non-Salem set's D_F is dominated by the behavior of relatively few and isolated Fourier coefficients. Their values are of no present interest in physics. Thus, D_F is of interest only when it reduces to D, in which case it provides an alternative way of calculating D. Otherwise, it provides a bound for D, but this bound is too loose to be useful and may be extremely difficult to evaluate. For further comments about so-called dimensionalities that are overly affected by arithmetic, see FGN, p. 362, paragraph 2.

Second Example: *The Mass Exponent in a Random Fractal Set.* An important property of fractal dimensionality is that, given a fractal point set with the fractal dimensionality D, the mass contained within a ball of radius R centered on this set is $M(R) \propto R^D$. This role of dimensionality is so essential that, whenever a power law rules the main-radius relation, one is tempted to say that its exponent is yet another generalized (or anomalous) dimensionality.

For example, the anomalous mass exponent of the random fractals that FGN p. 123 advances as models of percolation clusters, is *not* a *fractal* dimensionality. One may argue about the definition of fractal dimensionality but it must remain a property to be attached to specified point sets, not to random sets or other ensembles of sets.

In the early days of the use of fractals in the study of percolation clusters,[7] it has been asserted that, when there are alternative ways of

measuring the mass in a sphere of radius R, several values can be assigned to the fractal dimensionality. This conclusion gave rise to the feeling that fractal geometry is marred by indeterminacies and resulting "fuzziness." In fact, D is perfectly well defined for each of the sets involved in percolation. The seeming fuzziness was merely due to the physicists' custom of neglecting (or spurning) direct geometry. To dwell exclusively on indirect descriptions via analytic properties seems to put many different exponents on the same level. As stated in the introduction, geometry is essential and must revive.

Third Example: *The Diameter Exponent in Trees.* At points where trees branch out (FGN, Chapter 17), the branch diameter d on the root side and the branch diameters d' and d'' on the leaf side are often linked by a relation $d^{\Delta} = d'^{\Delta} + d''^{\Delta}$. Some authors call Δ a dimensionality and I briefly called it a paradimensionality, but I took this term back because Δ is surely not a fractal dimensionality. The same remarks apply to the Besicovitch and Taylor exponent (FGN, "warning" at the end of p. 259).

3.1.2. A Basic Qualification to the Basic Rule: Many Problems in Physics Involve *More Than One* Fractal Set. This paper has or will consider several classes of examples.

First Class of Examples (*Sections* 1.1.4 *and* 1.1.5). In statistical physics, the study of percolation involves the clusters at criticality, but also their boundaries and other sets such as the multiply connected ring portions (taken either singly or in various combinations). An even more elementary example is that of Brownian motion, with its trails, zerosets, hulls (FGN, p. 243), and the like.

Second Class of Examples (*Section* 3.2). Increasingly, for example in the study of turbulence and of strange attractors, one has to deal with fractal *measures*. By contrast, it is necessary to become aware of the fact that all fractals encountered in the earliest applications were compact fractal sets. The mathematical notion of compact set is thereby acquiring a concrete physical meaning, because the full specification of a fractal measure is necessarily equivalent to an infinity of distinct compact fractal sets.

Third Class of Examples (*Section* 3.3). Numerous fractal sets of interest decompose naturally into combinations of compact fractal sets, whose individual "fractal eigendimensionalities" may be of interest.

Fourth Class of Examples (*Section* 3.4). Different studies of the same fractal may imbed it in spaces endowed with different notions of distance, e.g., Euclidean distance or graph distance. These sets should be considered as distinct from each other.

3.1.3. Converse of the Remarks in Section 3.1.2. Up to now, whenever the impression has arisen that the application to the same

problem of several different definitions of D yields distinct values (and when no error was made), the different definitions have always turned out to be in fact relative to *different compact sets* implicit in the same original problem.

Often, these sets had previously escaped explicit attention, which justifies after the fact the labor invested in comparative testing.

When faced with fresh instances of conflicting estimates for the fractal dimensionality D, I seek to eliminate the generalized dimensionalities that are not fractal, and undertake a search for sets to which the different genuinely fractal estimates might be allocated as D's. This reaction was of course founded upon a belief in the *absolute primacy of geometry over analytic refinement*. This belief is constantly encouraged by the fact that, thus far, every would-be D that cannot be allocated as the fractal dimensionality of any set has indeed turned out *not* to be fractal, as explained in 3.1.2.

3.2. Fractal Measures, and a Fully Studied Example of Their Representation as "Composites" of Compact Fractal Sets

One concrete aspect of the complications and the opportunities implicit in the assertion in Section 3.1.3 first manifested itself between twenty and ten years ago, when my study of noise introduced into physics diverse mathematical objects I now call *fractal measures*. In early investigations, this notion did not have to be explicated, because it was in one-to-one correspondence with the notion of *compact fractal set*. One example was the derivative of the Devil's staircase, which is a measure that vanishes outside of a compact fractal set, namely, the Cantor dust, and is distributed on this set in "fractally homogeneous" fashion. I call such measures "absolutely intermittent." I used them long ago to provide the proper geometry for "absolutely intermittent noises" (2.2.2), and soon afterwards for galaxy distributions and for the Novikov–Stewart model of the "absolute intermittency" of turbulence. More interesting are the "relatively intermittent" fractal measures, which I first used to provide the proper geometry for "relatively intermittent noises" (2.2.3) and soon afterwards for a corrected form of the Kolmogorov model of "relative intermittency."[25,26,27,28] They have attracted the attention of excellent mathematicians[29,30] and are of interest in meteorology.[31] Now, some facts about fractal measures have been rediscovered by other authors,[32,33] and they are also used to model those fractal ("strange") attractors to which fractal sets[34] are not applicable.

3.2.1. A Vivid Example of Fractal Measure: Gold Mines. In my interpretation and elaboration of the de Wijs model of the distribution of

minerals (FGN, p. 376), *every* square on the surface of the Earth carries *some* gold. However, the overwhelming bulk of the gold (arbitrarily close to 100%) concentrates within a small compact fractal set on the surface of the Earth. And if one is only interested in gold of high and increasing purity, one must restrict one's search to increasingly smaller compact fractal sets. Each of these sets can be approximated by a fractal whose fractal dimensionality decreases with increases in the desired purity.

A limit case of this model allows all of the gold to be restricted to a fractal whose dimensionality is < 2, but of course above the dimensionality of the set where gold is concentrated.

Clearly, a fractal measure is much more involved mathematically than a compact fractal set. To put it simply, a set is identical to its indicator function, which is equal to 1 within this set and to 0 outside, while a measure is identical to a much more general function, and an increasingly detailed description of a fractal measure demands the introduction of an increasing number of fractal sets, each with its own fractal dimensionality. Reference 1, Chap. 9, and FGN, pp. 375–381 covers my own work circa 1974, and refers to the work of Besicovitch and other older mathematicians, and of J. Peyrière, J. P. Kahane, and other mathematicians since 1974. References 32 and 33, which largely overlap with old work, have stimulated wide interest. However, Ref. 32 asserts that a fractal measure has *two* dimensionalities, and chooses to apply the term "fractal" to only one of them. This assertion has no merit. The two quantities in question are the fractal dimensionalities of different sets implicit in the same fractal measure. Reference 33—which I find difficult to follow in detail—does not distinguish between fractal sets and fractal measures; it rediscovers the continuous infinity of exponents in Refs. 26 to 30, and labels many of them "generalized dimensionalities" without attempting to relate them to the fractal measure's geometry and without obtaining their values explicitly, and does not investigate the invariant fractal measure I had introduced.

Besicovitch Set. A. S. Besicovitch and his students made profound studies of the set of those points on $[0, 1]$ that have an "abnormal" decimal development, namely, of points for which the relative numbers of occurrences of the "decimal" g in the base b tends to the limits $p_g t$ that are not all equal to $1/b$. This deserves to be called the Besicovitch set. They proved that its Hausdorff–Besicovitch dimensionality is $D_{\mathrm{HB}} = -\sum p_g \log p_g$. On the other hand, this set is self-similar with a similarity dimensionality equal to $D_{\mathrm{SS}} = \log b / \log b = 1$. Hence $D_{\mathrm{HB}} \neq D_{\mathrm{SS}}$. This discrepancy is very peculiar *a priori*, but is put in perspective when it is realized that the closure of the Besicovitch set, i.e., this set plus its limit points, is the interval $[0, 1]$. Thus, the Besicovitch set is not compact, and D_{SS} picks up the fractal dimensionality of its closure.

Roughening Interpolation. The Besicovitch Measure. The Besicovitch set is associated with the Besicovitch measure. It could be introduced to physicists as a special case of the renormalization process in 3.2.3, but the usual construction is different and proceeds by a form of "roughening interpolation." (A recent expository article[35] is worth pointing out.) One starts with a uniform distribution of a unit mass on $[0, 1]$, then one redistributes this mass so that the successive subintervals of length $1/b$ carry the masses p_g adding to 1. Then the mass in each successive subinterval of length $1/b^2$ is again redistributed in the same ratios.

Asymptotically, the mass in the interval $[0, t]$ is a nondecreasing function $M(t)$. It has a zero derivative at almost all points t, but every interval $[t', t'']$, however small, contains *some* mass. One may label it an "eroded Devil's staircase." Its increase occurs entirely on the above-defined Besicovitch set. Historically, the construction may have been touched upon by Lebesgue, but it deserves to be named after Besicovitch, who spent a lifetime studying it. For a brief moment, I had thought it was my discovery, and the current interest in fractals leads to parts of its theory being rediscovered again under diverse curious labels. (The worst was "probabilistic fractal," apparently inspired by the fact that the letter p_g seems to denote a probability; I dissuaded its author from using it anymore, but some withdrawn preprints have lasting effects.)

Measures and Fractal Measures. A measure is simply the differential of a nondecreasing function $f(x)$ that is not differentiable. One writes the measure as $df(x)$, but one always examines it through integrals; whenever possible, these integrals are taken over intervals. For an ordinary differential, $\int_u^{u+\varepsilon} df(x) \propto \varepsilon$, but for a general measure, $\int_u^{u+\varepsilon} df(x)$ is not $\propto \varepsilon$. For many measures of interest in mathematics, and now also for many more measures constructed for the purposes of physics, $\int_u^{u+\varepsilon} df(x) \propto \varepsilon^{\beta}$ with the exponent β a function of u. In many cases, β takes the same value for almost all values of x; an example is white noise, which is the measure differential of Brownian motion. Very tentatively, we shall say that such measures are *fractal measures*. Much more can be said to comment and elaborate this definition, but this is not the proper place to do so.

3.2.2. The Fixed-Point Random Variables and Random Fractal Measures Introduced by Mandelbrot[26,27]: Background and 1974 Construction. Some background from little-known probability theory is useful. Let $X(n_1, 1)$ be a semi-infinite array of independent and identically distributed random variables in E-dimensional Euclidean space, the index n_1 being written as E integers in the base b. Adding all the $X(n_1, 1)$ that have the same index n_1 except for the last digit, creates an array $X^*(n_1, 2)$. The index n_2 is obtained by chopping off the last digit, and the unit of

space becomes b times larger for $X^*(n_2, 2)$ than for $X(n_1, 1)$. The next step "renormalizes" $X^*(n_2, 2)$ by forming the array $X(n_2, 2) = AX^*(n_2, 2) + B$. By repeating on $X(n_2, 2)$ the same operation of renormalization, one creates a twice renormalized array $X(n_3, 3)$. Long ago, in the 1920s, Paul Lévy defined the *semistable random variables* as being the variables one can obtain (with suitable choices of A and B) as the fixed points of this renormalization. He showed that the logarithm of the Fourier transform of the probability distribution must be of the form $-cu^{\alpha}P(\log u)$, where $P(x)$ is a periodic function of x, of period equal to $\log b$. In order for a random variable to be semistable for every value of b, $P(x)$ must be a constant, and the random variable is best called Lévy stable. The Lévy-stable fixed-point random variables (r.v.) are very important in the theory of fractals (e.g., Section 2.3.2), but the semistable fixed-point r.v. have drawn very little attention, because of the indeterminancy and complications brought in by their dependence on an arbitrary function $P(x)$.

After these preliminaries, let us return to fractal measures. Mandelbrot[26,27] has discovered that in the application to noise and turbulence it is not only inevitable but essential to introduce a somewhat analogous process of renormalization. And the somewhat analogous (though different) indeterminacy and complication present in the resulting random variables are concretely very important. The key ingredient in this more general renormalization is to replace ordinary addition by randomly weighted addition. The weights are a semi-infinite array of independent identically distributed r.v. with row index n and column index i, $W(n, i)$. Now we start with $X(n_1, 1) \equiv 1$, and the first step of renormalization is to form the array $X^*(n_2, 2) = \sum W(n_1, 1)X(n_1, 1)$ with the sum carried over the indexes n_1 of the form n_2 followed by an integer between 0 and $b - 1$. Of the many classes of W examined in Ref. 26, the simplest, and only one that has been studied further,[28,29,30] is the class characterized by $W \geqslant 0$, and $\langle W \rangle = 1$. The proper second step in renormalization is then $X(n_2, 2) = b^{-E}X^*(n_2, 2)$, and $\langle X(n_k, k) \rangle \equiv 1$.

The first object of study is, then, the fixed-point random variable $X = \lim_{k \to \infty} X(n_h, k)$ that is invariant under renormalization. The second object is to interpolate $X(n_k, k)$ into a random function $X(n_1, k)$ and to study the contributions to $X(n_1, k)$ from the addends $X(n_1, k) = X(n_1, k) - X(n_n - 1, k)$ originating in the little cubes.

3.2.3. The Kolmogorov–Yaglom Mass Exponents of X, $M(h) = Eh - \log_b \langle W^h \rangle$. The most elementary characteristic of $X(n_1, k)$ is the sequence of its moments. A quick argument shows that $\langle X(n_1, k) \rangle \propto n_1^E$ and, more generally, that $\langle X^k, (n_1, k) \rangle \propto n_1^{M(h)}$, where $M(h) = Eh - \log_b \langle W^h \rangle$ is called mass exponent of order h. However, this quick argument is not exact: I have pointed out that it holds only if $\langle X^k(n_1, k) \rangle$ is

finite, which is *not* necessarily the case. In fact, it will result from 3.2.6 that $\langle X^k(n_1,k)\rangle < \infty$ only if $M(h) > E$.

Several of the $M(h)$, both for integer and noninteger h, have direct physical meaning. For example, $M(2)$ rules the correlation properties, and I showed[28] that $M(2/3)$ rules the correction to the $5/3$ exponent in the Kolmogorov spectrum. Thus, knowledge of $M(2)$ and $M(2/3)$ suffices for the second-order properties. But a full specification of X requires a full specification of W. In the usual cases (as identified by the so-called "moment problem" of analysis), it suffices to know $\langle W^h\rangle$ for integer h.

As implied in the second example of 3.1.1, these mass exponents *are not fractal dimensionalities*, and to call them *generalized dimensionalities* would serve no purpose.

3.2.4. The Critical Function $C(h)$. It will be seen in 3.2.5 and 3.2.6 that all the fractal dimensional properties of X are ruled by W through a function I introduced indirectly in 1974 and now write in the form of the "critical function" $C(h) = \log_b\langle W^h\rangle/(h-1)$. For $h = 1$, by the L'Hospital rule, $C(1) = \langle W\log_b W\rangle$, which is > 0 (because $x\log x$ is a cap-convex function hence $\langle W\log W\rangle > \langle W\rangle\langle\log W\rangle = 0$). The function $C(h)$ is defined for all real h from $-\infty$ to $+\infty$; it is continuous and nondecreasing; it is cup convex; $C(\infty)$ is finite if and only if W is bounded; $C(-\infty)$ is finite if and only if $1/W$ is bounded; if $\langle W^h\rangle = \infty$ for $h > h^+$ (or for $h < h^-$), then $C(h) = \infty$ for $h > h^+$ (or for $h < h^-$).

Case Where $C(h)$ is Constant. The simplest case is when $W = 0$ with the probability $1-p$ and $W = 1/p$ with the probability p, therefore $C(h) \equiv -\log_b p$. This case is called "absolute curdling," as explained in FGN, p. 210. Everything of interest depends on the scalar parameter p. The reason is that the fractal measure of absolute curdling concentrates in fractally homogeneous fashion on a Cantor-like random dust, a compact fractal set. This set's fractal dimensionality is $-\log_b p$, and it determines everything of interest.

Cases Where $C(h)$ Varies with h. The corresponding curdling is "weighted" (FGN, p. 378), and the "parameter" that determines the fractal properties is not one number, but the whole function $C(h)$, i.e., an infinity of numbers. Fortunately, the fractal properties can be ranked in order of importance, therefore most problems involve only $C(1)$ and perhaps a few other values of $C(h)$.

Alternatively, one may approximate a complicated $C(h)$ by either of several simple functions. The very simplest is a constant. To write $C(h) \equiv C(1) = \langle W\log_b W\rangle$ amounts to approximating weighted curdling by the absolute curdling corresponding to $p = b^{-C(1)}$.

The second simplest approximation is $C(h)$ proportional to h. It is exact when W is lognormal and $\langle W\rangle = 1$, hence it is best to write

$C(h) = \mu h/2$, in order to match the notation Kolmogorov uses in his important (though self-contradictory) lognormal model of intermittency.

Again, however, a full knowledge of W requires all the integer order value of $C(h)$.

3.2.5. $h = 1$ and the Critical Dimensionality $C(1) = \langle W \log_b W \rangle$—Hence the Fractal Codimensionality $1 - C(1)$—of Mandelbrot.[26,27] The first task I set up and solved in 1974 is to derive the fractal dimensionality of the point set which carries the bulk of the local contributions. In the language of 2.1, the latent value D_L was found to be $D_L = C(1) = E - \langle W \log_b W \rangle$. When $D_L < 0$, the actual D is 0. This possibility is easy to understand, as long as $\langle W \log W \rangle < \infty$. It suffices to carry out the same renormalization argument, using the same weights but working in a Euclidean space with dimensionality $E^+ > \langle W \log_b W \rangle$. The set of concentration of the resulting fixed point distribution is of dimensionality $E^+ - \langle W \log_b W \rangle$. Next, one intersects by the original space of dimensionality E, and one finds that the intersection is almost surely empty, its degree of emptiness being measured by D_L. Thus, $E = C(1) = \langle W \log_b W \rangle$ is a critical dimensionality associated with the weights W. This $C(1)$ may be an integer, for example if $C(h) = \mu h/2$ with μ an even integer. But in the general case, $C(1)$ need not be an integer. It might have been an ill-defined "continuous dimensionality," however, 3.2.9 will show that it can be interpreted rigorously as the fractal dimensionality of a fully specified fractal point set.

[In probabilistic terms, $D_L < 0$ implies that $\lim_{k\to\infty} X(n_k, k)$ is degenerate, i.e., almost surely equal to 0. One deals with one of those cases beloved by mathematicians, when $\lim_{k\to\infty} \langle X(n_k, k) \rangle = 1$ but $\langle \lim_{k\to\infty} X(n_k, k) \rangle = 0$. Physicists do not expect such cases to be of concrete importance, but the present occurrence is not only important but very simple.]

3.2.6. $h > 1$ and the Critical Dimensionalities $C(h) = \log_b \langle W^h \rangle / (h - 1)$ of Mandelbrot.[26,27] The second task I set up and solved in 1974 is to determine the integer or noninteger moments of $X = \lim_{k\to\infty} X(n_k, k)$, assuming that $D_L = 1 - C(1) > 0$ so that the limit random variable X is nondegenerate. The key findings, including those of J. Peyrière and J. P. Kahane, are these. When the equation $C(h) = E$ has a root > 1, to be denoted by $h = \alpha(E)$, then $\langle X^k \rangle < \infty$ for $h < \alpha(E)$ and $\langle X^h \rangle = \infty$ for $h > \alpha(E)$. This function $\alpha(E)$ is continuous and increasing in E. When $C(1) = E$, one can write $\alpha(E) = 1$. When $C(h) = E$ has no root $\geqslant 1$, then $\langle X^h \rangle < \infty$ for all h, and one can write $\alpha(E) = \infty$; for example if W is bounded so that $C(\infty) < \infty$, then $\langle X^h \rangle < \infty$ for all $E \geqslant C(\infty)$. Observe that $C(h) - E < 0$ means $\log \langle W^h \rangle - E(h - 1) < 0$, therefore is equivalent to the condition $M(h) > E$ announced in 3.2.3.

Now use the same W to renormalize in spaces of varying E. In absolute curdling $C(h) = -\log p$, hence $E > -\log p$ leads to $\alpha = \infty$ and $E \leqslant -\log p$ leads to $D_L \leqslant 0$. In weighted curdling, there corresponds to each $h \neq 1$ the critical dimensionality $C(h)$ such that $\langle X^h \rangle < \infty$ if and only if $E > C(h)$. The meaning of a noninteger $C(h)$ is, again, explained in 3.2.9.

3.2.7. The $C(h)$ as the Latent Fractal Codimensionalities of Suitable Level Sets of Δx. From 3.2.5, almost all of the variation of X occurs on a set of latent codimensionality $C(1)$. Similarly, one finds that almost all the contributions to $\langle X^h \rangle$ come from a set of latent codimensionality $C(h)$.

3.2.8. $h = 0$ and the Critical Dimensionality $C(0)$. Weighted curdling includes absolute curdling as a special case. More generally, weighted curdling allows for either $W > 0$ or $W \geqslant 0$. When $W > 0$, the fractal measure is relatively intermittent over the whole of $\mathbb{R}^E$. When $W \geqslant 0$ the fractal measure is relatively intermittent over a subset of $\mathbb{R}^E$ that is obtained by absolute curdling, and whose latent fractal dimensionality is $\log_b[b^E \langle W^0 \rangle] = E + \log_b \langle W^0 \rangle = E - C(0)$. [Terminological aside: Refs. 32 and 33 restrict the use of the term "fractal" to the dimensionality $E - C(0)$, whereas we saw that the measure generated by W implies an infinity of other fractal point sets of interest, each with its fractal dimensionality.]

As in 3.2.5 and 3.2.6, there is a critical E equal to $C(0)$. When $W > 0$, hence $C(0) = 0$, this critical E is 0. When $C(0) > 0$, the significance of the condition $E > C(0)$ results from familiar facts about fractals, FGN, pp. 213–214.

3.2.9. Interpretation of the Noninteger Values of E. A Form of Subordination as Applied to Fractal Measures. It is fruitful to consider an alternative presentation of the results in 3.2.8 relative to the special case when $W \geqslant 0$ and $E > C(0)$. The first step is to introduce a random variable W^* for which the value $W = 0$ is disallowed and the probability $\Pr(W = 0) = 1 - \langle W^0 \rangle$ is redistributed among all $W > 0$. This new weight W^* is such that, for every interval I, one has $\Pr(W^* \in I) = \Pr(W \in I) / \Pr(W > 0)$.

Using this W^*, weighted curdling using W on the whole $\mathbb{R}^E$ can be viewed as equivalent to weighted curdling using W^* on a subset of $\mathbb{R}^E$ whose fractal codimensionality is $C(0)$. In this sense, the inequality $C(0) > 0$ should lead us to deemphasize the dimensionalities $E - C(h)$, and to feature instead the dimensionalities $E - C(0) - C(h) = E + \log\langle W^0 \rangle - \log\langle W^h \rangle = E - \log[\langle W^h \rangle / \langle W^0 \rangle] = E - C^*(h)$.

Conversely, consider a W such that the critical dimensionality $C(h)$ is not an integer. One can "lift up" $C(h)$ to any desired integer value above $C(h)$, by choosing new weights W^+ that can vanish, whose probability of being positive is $\langle W^+ \rangle < 1$, and such that $\log[\langle W^h \rangle \langle W^+ \rangle]/(h-1)$ is the desired integer. When $C(h) = \mu h/2$ with μ an odd integer, one W^+ suffices to ensure that all the $C(h)$ become integers simultaneously.

3.3. Fractals with a Nontrivial Eigen-Decomposition

The complications and the opportunities implicit in the assertion made in the title of 3.1 find a homey example in certain Cantor dusts. Start with all the (real) numbers in $[0, 1]$ represented in the counting base $b > 3$. The reals whose representation *fails* to include 2 form a Cantor dust C' of dimensionality $\log(b-1)/\log b$, and the reals whose representation fails to include either 2 or 4 form a Cantor dust C'' that is contained in C' and has the dimensionality $\log(b-2)/\log b$. What about $C = C' - C''$? This is a fractal dust whose Hausdorff dimensionality D is the same as that of C', but clearly D gives an incomplete description of C. For certain uses, the dimensionality of C'' may also be important.

Furthermore, certain methods of evaluating the fractal dimensionality involve a matrix that Ref. 16 calls transfer matrix of a fractal, TMF. Let λ_1 be the leading eigenvalue of the TMF, the other eigenvalues being λ_i. The expression $\log\lambda_1/\log b$ happens to coincide with the dimensionality of C'. For C' or C'' taken separately, there are no other nontrivial eigenvalues. But for $C' - C''$ there is a second nontrivial eigenvalue λ_2 that happens to be $b-2$, hence the dimensionality $\log(b-2)/\log b$. Clearly, the eigendimensional sequence can serve to distinguish between sets that have identical overall fractal dimensionalities.

Given the above C, the above complication could be predicted, because the set C does not include its limit points; it is not closed hence is not compact. The smallest compact set including C is C', and Hausdorff's form of fractal dimensionality "seeks out" this compact set. Then the compact set C'' is defined as a difference, hence the decomposition $C = C' - C''$ could be reconstituted even if one did not know it in advance.

3.4. Fractal Graphs, Dimensionality When the Shortest Distance is Measured Along Edges

FGN concerns fractals imbedded in Euclidean space, the underlying distances being Euclidean, except that in a few cases they are measured along circles. On the other hand statistical physics often deals with interactions that propagate solely along the bonds in a graph that is not imbedded in any Euclidean spaces. This graph may be constructed recursively, as for

example in the work of Nelson and Fisher[36] and of Berker and Ostlund.[37] However, the notion of fractal extends readily to other metric spaces. In particular, as pointed out by Suzuki,[38] the notion of similarity dimensionality extends to graphs, the graph distance between nodes being defined (as usual) as the minimum of the number of links along the paths joining these nodes in the graph. (Suzuki had originally referred to this similarity dimensionality as an *interpolated topological dimensionality*, but recognized that this notion is in fact not topological. It is metric but non-Euclidean.) The scope of the graph similarity dimensionality extends beyond fractals, since it is also defined for Cayley trees (for which it is infinite) and for other graphs that cannot be interpolated.

Many truncated fractals in Euclidean space can also be viewed as graphs and endowed with the graph distance. In the example of the Sierpinski gasket, the ratio of graph to Euclidean distance is contained between two finite and positive numbers, so their dimensionalities defined via these alternative distances are identical. But other truncated Euclidean space fractals behave differently. Consider for example a Koch curve whose generator is a broken line without branch or ring. The fractal dimensionality based on the graph distance is 1, independently of the generator. Therefore, all such fractal Koch curves are equivalent when viewed as fractal graphs; this expresses the fact that topologically they are straight lines. Consider also the diamond fractal lattice, which can be endowed either with graph distance, as in Berker and Ostlund,[37] or with Euclidean distance. The graph fractal dimensionality is 2, and the Euclidean fractal dimensionality can take any value above 2.

3.4.1. Random Walk on a Fractal Graph. The graph's fractal dimensionality D and the walk's latent fractal dimensionality $\alpha = 1/H$ can be defined as in Section 2.4. When the distance is Euclidean, the values of D and of α both depend on the actual imbedding that is chosen. But the ratio D/α is independent of the imbedding. Furthermore, it takes the same value as when the graph distances are used. This should have been expected, because the ratio D/α is the codimensionality of the instants on the time axis when the walk recurs to its point of departure. On the time axis, the metric offers no choice: time is always endowed with Euclidean distance.

3.4.2. Solution of a Paradoxical Conflict Between Two Evaluations of D. The fractal dimensionality D is often evaluated via the relation "mass $M(R)$ in a sphere of radius $R \propto R^D$." Actually, this is not the primary definition of D and the relation $M(R) \propto R^D$ is *not* expected to hold for *all* origins on the fractal, only for "almost all origins." This clause "almost all" is almost always disregarded by physicists. Usually, there is no

harm, but there are conspicuous examples where disregarding it leads to a paradox.

One such example is the diamond fractal lattice. To be specific, endow it with Euclidean distance, taking as initiator the segment [0, 1] and as generator the square of diameter [0, 1]. The embedding space dimensionality must be sufficiently high, but otherwise does not matter. The similarity dimensionality is $D = \log 4/\log\sqrt{2} = 4$. Now compare the masses within spheres of radii $2^{-k/2}$ centered on 0: it is easy to see that these masses scale like R^2 instead of R^4. The same holds if the origin is any other vertex of the generator of diameter [0, 1].

Here is the explanation. The above origins, and all the other conspicuous points, are generated by a finite number of iterations, and after that remain pinned down. They constitute the "rational points" of the lattice, and are denumerable, while the irrational points obtainable only by infinite interpolation are nondenumerably infinite. Hence, the rational points are negligibly few compared to the irrational points. An origin chosen at random will be an irrational point. An irrational point is buried in an infinite sequence of approximating boxes whose sides downscale like $\sqrt{2}$ and whose contents downscale like 4. For these points, $D = 4$, as it should. When a point is generated by a finite but large number of stages of construction, in a small box around it, $M(R) \propto R^2$ but in a large box $M(R) \propto R^4$.

An analogous issue arises with the notion of order of ramification (FGN, Chapter 14), when it is defined in the mathematicians' fashion, by interpolation. For the Sierpinski gasket, the order of ramification is 3 for most points, but it is 4 for the rational points, which are the only ones seen on approximate illustrations. The order of ramification defined by extrapolation is always 3.

4. PART FOUR: THE DEFINITION OF FRACTALS

The reader must have noticed that, while the notion of fractal dimensionality was being "opened up" on its latent form (Section 2.1), and while its unicity was being proclaimed (Section 3), this paper advanced no definition of either "fractal" or "fractal dimensionality." Let me elaborate —though I do know that having coined these terms gives me no permanent control of their use. Today, I use the term "fractal dimensionality" *generically*, as equally applicable to numerous, but not all, specific definitions of anomalous dimensionality, and I try *not* to have to define "fractal." Yesterday, in many places in Refs. 2 and 3, I had made half-hearted attempts to use *fractal* dimensionality as synonym to *Hausdorff–Besicovitch* dimensionality, D_H, and I had given the "tentative definition" of a fractal set, as being a set for which $D_H > D_T$. Today, I think both attempts were

misled. Both had been carefully avoided in my earliest essay[1] (in French). I regret having later been swayed by the notion that the short-term purpose of gaining acceptance for the study of fractals demanded such definitions, at least tentative ones. Furthermore, I have sometimes let my pen slip and describe $D_H > D_T$ as the definition of *a fractal* instead of being precise and referring to a *fractal set*. The more casual wording did not allow for the introduction of *fractal measures*, which (as described in Section 3) I had been using since 1974 at least, but which had not come to the fore until recently. All that brought no durable harm, but has tended to bring confusion to the physicists, and to be disregarded by the mathematicians. Anyhow, the definitions' short-term purpose has by now been fully achieved; they may have been a useful crutch for a while, but have outlived their usefulness.

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